Lecture No. 4

Summary of the Conventional Galerkin Method

$$L(u) - p(x) = 0 \quad \text{within } \Omega$$

$$S(u) = g(x) \quad \text{on} \quad \Gamma$$

- Construct an approximate solution:

$$u_{app} = u_B + \sum_{i=1}^{N} \alpha_i \phi_i(x)$$

- For the conventional Galerkin method:

  - u_{app} satisfies the specified b.c.'s

$$S(u_B) = \overline{g}$$

  - ϕ_i are the trial functions from a complete sequence, sufficiently differentiable, and satisfying the homogeneous form of the boundary conditions

$$S(\phi_i) = 0, \quad i = 1, N$$

- We must satisfy the b.c.'s exactly for admissibility. This is very difficult for 2-D irregular boundaries. Let's try to relax this very strict requirement on b.c.'s.

- Solve for the unknowns by enforcing a set of orthogonality conditions:

$$< \mathcal{E}_I, w_j > = 0, \qquad j = 1, \ldots, N$$

For Galerkin (test and trial functions are the same)

$$w_j = \phi_j$$

- Thus

$$< (L(u_B) - p), \phi_j > \; + \sum_{i=1}^{N} \alpha_i < L(\phi_i), \phi_j > = 0, \qquad j = 1, N$$

- We must also satisfy the functional continuity requirements such as the $L(\phi_i)$ is defined (second part of admissibility requirements). ***However we note that the trial and test functions have <u>different</u> functional continuity requirements.*** Let's try to balance this requirement since for Galerkin methods the trial and test functions are the same.

- When the operator is self adjoint, the conventional Galerkin method gives you a symmetrical matrix:

$$< L(\phi_i), \phi_j > = < L(\phi_j), \phi_i > \; + \textit{boundary terms}$$

- When L is self adjoint and positive definite you get a symmetrical matrix which is also positive definite

Weak Formulations

The philosophy of weak formulations is to relax b.c. enforcement and functional continuity requirements for the basis functions.

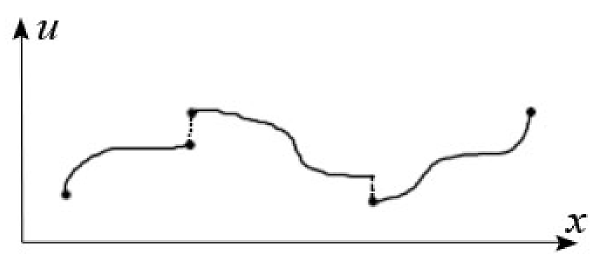
We can accomplish these things through the integration by parts procedure.

Classification for Degree of Continuity of a Function: Sobelev Spaces:

$W^{(0)} = L_2$ function space:

Functions which are discontinuous but finite.

$$\|u\|_0 = \iint_V u^2 dV < \infty$$

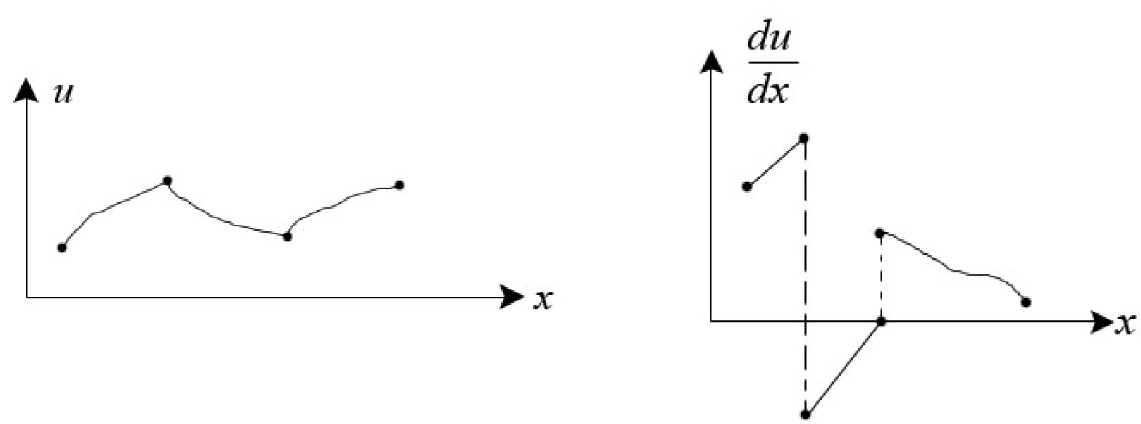


$W^{(1)} = C_0$ function space:

Represents the set of functions for which the norm of the function and the norm of the first derivative are bounded:

$$\|u\|_1 = \iint_V \left[(u)^2 + \left(\frac{du}{dx}\right)^2 \right] dV < \infty$$

In the $W^{(1)}$ space, the 1st derivative is bounded and the function is continuous. Therefore there is zero continuity of the first derivative and only the function is continuous.

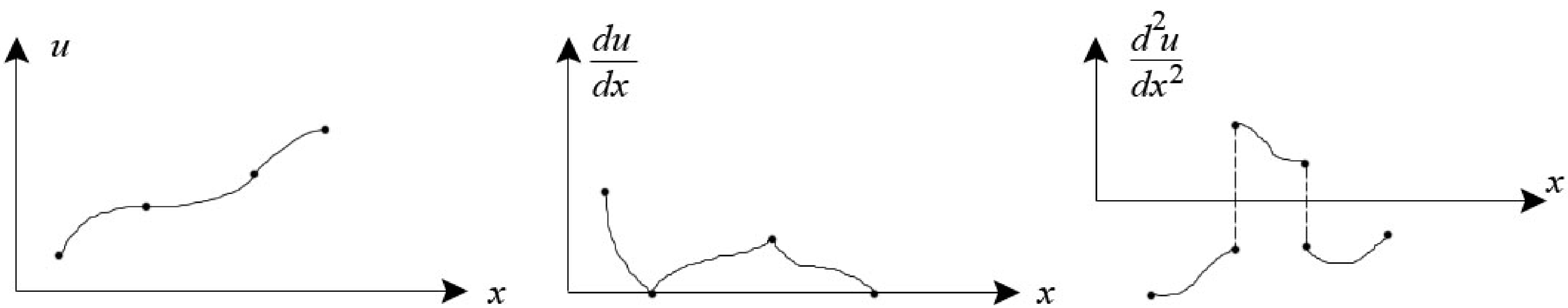


$W^{(2)} = C_1$ function space:

Represents the set of functions for which the norm of the function, the first derivative and the second derivative are bounded:

$$\|u\|_2 = \iint_V \left(u^2 + \left(\frac{du}{dx}\right)^2 + \left(\frac{d^2u}{dx^2}\right)^2 \right) dV < \infty$$

Thus in $W^{(2)} = C_1$ space, the first derivative and the function are continuous while the second derivative is bounded.



Weak Solutions

1. Weak solutions are solutions which are pieced together over various intervals and satisfy only a limited degree of functional continuity.

- Criteria for choosing spaces is that the ***Integrated must be finite***.

  e.g. For $L(u) = \frac{d^2u}{dx^2}$, we can't have the second derivative not being defined.

  Therefore we need at least $W^{(2)}$ space (i.e. the second derivative must be bounded).

  Note that so far we've really been working with C_∞ or $W^{(\infty)}$ space, i.e. completely continuous functions.

- In general we require $W^{(j)}$ space where *j* is the highest derivative.

2. For the weak solution we will also allow an error on the natural boundary in addition to the interior error. Thus

$$\varepsilon_I = L\left(u_{app}\right) - p(x)$$

$$\varepsilon_{B,N} = S_N\left(u_{app}\right) - g_N(x) \quad \text{on } \Gamma_N$$

Now we will constrain both the interior and natural boundary error through the orthogonality conditions:

$$< \varepsilon_I, w_j > + < \varepsilon_{B,N}, w_j > = 0$$

This is called the "fundamental" weak form.

Note that we still require the trial functions to satisfy the essential b.c.'s.

Symmetrical Weak form (optimum weak form)

The symmetrical weak form is an integrated by parts form of the weighted residual formulation such that the functional space requirements on the trial <u>and</u> test functions have been balanced.

- For the weak form, we required $W^{(j)}$ space where j was the highest derivative.
- By using an integrated by parts form of the weak weighted residual equation, we can lower the functional continuity requirements on the trial functions.
- However we are raising the differentiability requirements for the test functions. This is not a problem since the trial and test function are the same leading to a balanced formulation with equal functional space requirements on the trial and test functions.
- A second major benefit of the integration by parts procedure of the fundamental weak form is that we eliminate certain parts of our natural boundary error.

  (Recall that we defined that the natural b.c.'s were through an integration by parts procedure that that's why they can be eliminated again by this same process.)

Example

$$L(u) - p = \frac{d^2u}{dx^2} + u - x = 0$$

Specified essential b.c. $u = f \quad @\, x = 0$

Specified natural b.c. $\frac{du}{dx} = g \quad @\, x = 1$

Let:

$$u_{app} = u_B + \sum_{i=1}^{N} \alpha_i \, \phi_i$$

- We require that u_B satisfy only the specified essential b.c.'s. Therefore

$$u_B = f \;@\; x = 0$$

Thus the test or approximating functions must not only satisfy homogeneous essential b.c.'s:

$$\phi_i = 0 \;@\; x = 0 \qquad i = 1, N$$

- Since for the Galerkin method the test functions are identical to the trial functions (of the weighting functions are the same as the approximating functions):

$$\phi_i = w_i \qquad i = 1, N$$

- Therefore the weighting functions automatically satisfy the homogeneous essential b.c.'s

$$w_i = 0 \ @ \ x = 0 \qquad i = 1, N$$

- Now we shall allow for an interior error $\mathcal{E}_I$ and a natural boundary error $\mathcal{E}_B$:

$\mathcal{E}_I = \frac{d^2 u_{app}}{dx^2} + u_{app} - x \quad \Rightarrow \quad$ Weigh and integrate over the interior domain

$\mathcal{E}_{B,N} = \left(g - \frac{du_{app}}{dx}\right) \quad \Rightarrow \quad$ Weigh and integrate over the natural boundary

$$\int_0^1 \left\{\frac{d^2 u_{app}}{dx^2} + u_{app} - x\right\} w_j dx + \left[\left(g - \frac{du_{app}}{dx}\right) w_j\right]_{x=1} = 0 \qquad j = 1, N$$

This represents the fundamental weak form and at this point $w_i \in L_2$ and u_{app} (or ϕ_i) $\in W^{(2)}$

- Integrating the fundamental weak form by parts:

$$\int_0^1 \left\{ (u_{app} - x) w_j - \frac{du_{app}}{dx} \frac{dw_j}{dx} \right\} dx + \left[\left(\frac{du_{app}}{dx} \right) w_j \right]_0^1 + \left[\left(g - \frac{du_{app}}{dx} \right) w_j \right]_{x=1} = 0$$

- Expanding out the boundary terms:

$$\int_0^1 \left\{ (u_{app} - x) w_j - \frac{du_{app}}{dx} \frac{dw_j}{dx} \right\} dx + \left[\left(\frac{du_{app}}{dx} \right) w_j \right]_{x=1} - \left[\left(\frac{du_{app}}{dx} \right) w_j \right]_{x=0}$$

$$+ \left[g w_j \right]_{x=1} - \left[\left(\frac{du_{app}}{dx} \right) w_j \right]_{x=1} = 0$$

- Note that $w_j \big|_{x=0} = 0$ since it satisfies the homogeneous b.c.'s and therefore:

$$\int_0^1 \left\{ (u_{app} - x) w_j - \frac{du_{app}}{dx} \frac{dw_j}{dx} \right\} dx + \left[g w_j \right]_{x=1} = 0 \qquad j = 1, N$$

u_{app} (i.e. ϕ_j) $\in W^{(1)}$ and $w_j \in W^{(1)}$.

Thus we must have a continuous function and the first derivative must be bounded.

- Thus we have reduced the functional continuity requirements for ϕ_j such that they are equal to those of w_j. This is a ***symmetrical formulation***.
- This is very desirable since for Galerkin methods since $\phi_j = w_j$.
- Therefore we have in effect lowered the required functional continuity to $W^{(1)}$ for the functions we will use. We have also simplified the b.c. treatment by not requiring ϕ_j (*equal to* w_j) to satisfy the natural b.c.'s in addition to simplifying the form of the boundary error term.

- Integrating by parts once more:

$$\int_0^1 \left\{ (u_{app} - x) w_j + u_{app} \frac{d^2 w_j}{dx^2} \right\} dx + \left[g w_j - u_{app} \frac{d w_j}{dx} \right]_0^1 = 0$$

Thus now:

$$u_{app} \in L_2 \quad \text{and} \quad w_j \in W^{(2)}$$

However this doesn't really help us since $\phi_j = w_j$, we still need a $W^{(2)}$ space requirement on ϕ_j.

Detailed Implementation of the previous example

$$L(u) - p = \frac{d^2u}{dx^2} + u - x = 0$$

b.c.'s

$$u = 0 \quad @ \quad x = 0$$

$$\frac{du}{dx} = 0 \quad @ \quad x = 1$$

- We developed the ***fundamental weak weighted residual formulation***:

$$< \mathcal{E}_I, w_j >_V + < \mathcal{E}_{B,N}, w_j >_{\Gamma_N} = 0$$

$$\Rightarrow$$

$$\int_0^1 \left\{\frac{d^2u}{dx^2} + u_{app} - x\right\} w_j dx + \left[\left(g - \frac{du_{app}}{dx}\right) w_j\right]_{x=1} = 0$$

We could now use this to substitute in our approximating functions which must:

i. satisfy ***relaxed*** admissibility
   - $\phi_i \in W^{(2)}$
   - Easier b.c.'s ϕ_i only has to satisfy the essential b.c.'s

ii. ϕ_i must be from a complete sequence

- However we can simplify the fundamental weak form by integrating by parts, which leads to the ***Symmetrical Weak Weighted Residual Form***

$$\int_0^1 \left\{ \left(u_{app} + x\right) w_j - \frac{du_{app}}{dx} \frac{dw_j}{dx} \right\} dx - \left[g w_j\right]_{x=1} = 0$$

This form requires $\phi_i \in W^{(1)}$ and has further simplified the boundary conditions.

- Let's now use the following approximation:

$$u_{app} = u_B + \alpha^{(1)}\, \phi_1(x) + \alpha^{(2)}\, \phi_2(x)$$

where

$$\phi_1(x) = x \quad \text{and} \quad \phi_2(x) = x^2$$

Evaluating the trial functions at the essential boundary $x = 0$

$$\phi_1(0) = 0 \quad \text{and} \quad \phi_2(0) = 0$$

Therefore the trial functions satisfy the b.c. requirements

- Thus u_{app} satisfies the "weakened" admissibility requirements for the symmetrical weak form:
  - u_B and ϕ_i satisfy the essential b.c.'s
  - ϕ_i only requires $W^{(1)}$ functional continuity
  - However we are still using $W^{(\infty)}$ for this particular example

  - Note that the natural b.c.'s are no longer necessarily satisfied and that

$$\left.\frac{du_{app}}{dx}\right|_{x=1} = \alpha^{(1)} + 2\alpha^{(2)}$$

does not necessarily satisfy the right hand side b.c. Also note that our u_{app} for this weak formulation is quite a bit simpler than that required for the conventional Galerkin method.

- Substitute u_{app} into our symmetrical weak weighted residual form:

$$\int_0^1 \left\{\left(\alpha^{(1)}x + \alpha^{(2)}x^2 + x\right)w_j - \left(\alpha^{(1)} + 2\alpha^{(2)}x\right)\frac{dw_j}{dx}\right\}dx - \left|gw_j\right|_{x=1} = 0 \qquad j = 1,2$$

In our case $g(x = 1) = 0$

- For $j = 1, w_1 = x$

$$\int_0^1 \left\{\left(\alpha^{(1)}x + \alpha^{(2)}x^2 + x\right)(x) - \left(\alpha^{(1)} + 2\alpha^{(2)}x\right)(1)\right\}dx = 0$$

- For $j = 2, w_2 = x^2$

$$\int_0^1 \left\{\left(\alpha^{(1)}x + \alpha^{(2)}x^2 + x\right)(x^2) - \left(\alpha^{(1)} + 2\alpha^{(2)}x\right)(2x)\right\}dx = 0$$

We now perform the integrations to obtain 2x2 system of equations:

$$\begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \cdots \\ \cdots \end{bmatrix}$$